Investigating Critical Phenomena in Axisymmetric Black Hole Formation with Numerical Relativity

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Overview

Critical phenomena in general relativity occurs when a generic system of mass and energy characterized by a parameter p is tuned to the threshold between gravitational collapse into a black hole and dispersion of the interacting matter. Due to the complexity of Einstein's equations, no analytical solutions exist in this regime, and thus critical phenomena can only be observed through numerical simulation. Choptuik's seminal work in this field [1] discovered three main aspects of critical phenomena: universality, echoing, and power-law mass scaling.

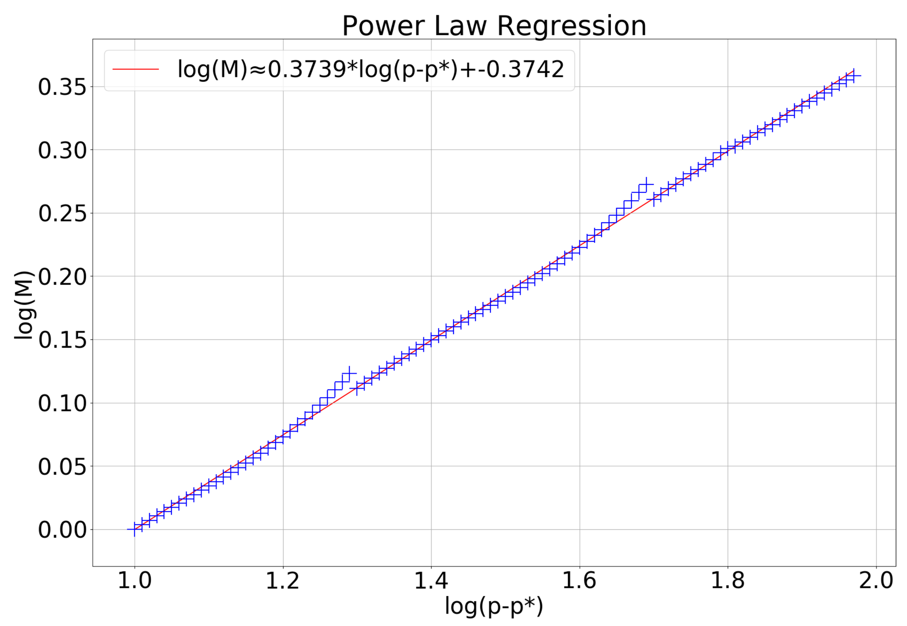

Figure 1: Demonstration of power-law mass scaling $M \propto |p - p*|^{\gamma}$. Initial data was a Gaussian scalar field.

While critical phenomena has been well-studied and tested for spherically symmetric systems of initial data, the same can not be said for less symmetric (and thus more physical) systems, which are complicated by additional degrees of freedom including the two polarization modes of gravitational waves. Indeed it is not even clear whether such systems exhibit the same properties as spherically symmetric critical phenomena, namely universality [2]. Prior work also predicts intriguing family dependent effects such as an infinite bifurcation of self-similar solutions for scalar field initial data, but was unable to confirm these effects due to computational constraints [3]. I aim to test these effects using recent improvements in numerical techniques, computational power, and a more stable formulation of Einstein's equations to achieve higher degrees of accuracy.

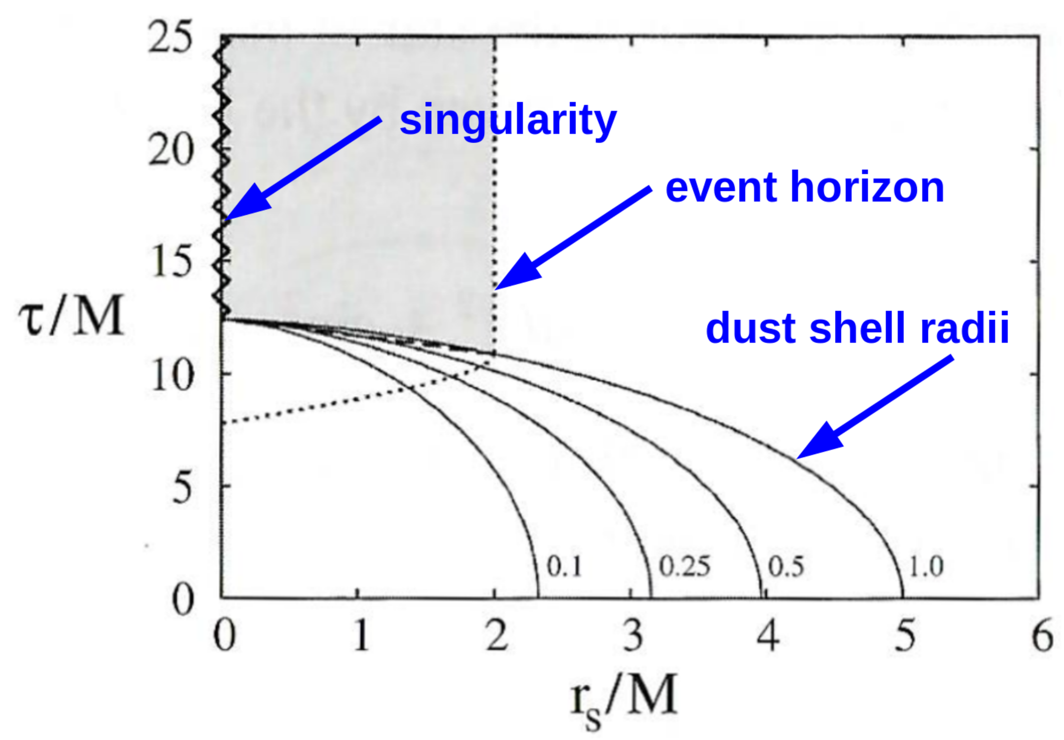

Figure 2: Example of cosmic censorship in the case of dust collapse. For typical initial data, an event horizon will always form in a region before a curvature singularity does. [4]

Here I present my implementation of a high order finite difference method with adaptive mesh refinement as well as results from applying this method to solve axisymmetric scalar field and gravitational wave initial data.

Motivation

Critical phenomena has drawn particular interest due to its implications for cosmic censorship. For typical systems, an event horizon will always form before a curvature singularity, but critical collapse enables arbitrarily large curvatures to form before an event horizon does, which appears to violate cosmic censorship.

Critical phenomena also has the potential to serve as a production mechanism for primordial black holes; a hypothetical class of black holes with significantly less mass than stellar black holes (as little as 10^{-8} kg). Such black holes would have been prime seeds for the modern supermassive black holes located in the centers of galaxies and are even considered to be a plausible candidate for dark matter. The mass scaling property of critical collapse enables arbitrarily small black holes to form, providing a physical mechanism by which primordial black holes may have been produced [5]. While they remain purely theoretical, the recent surge in observational black hole physics has lead to renewed interest in this hypothesis.

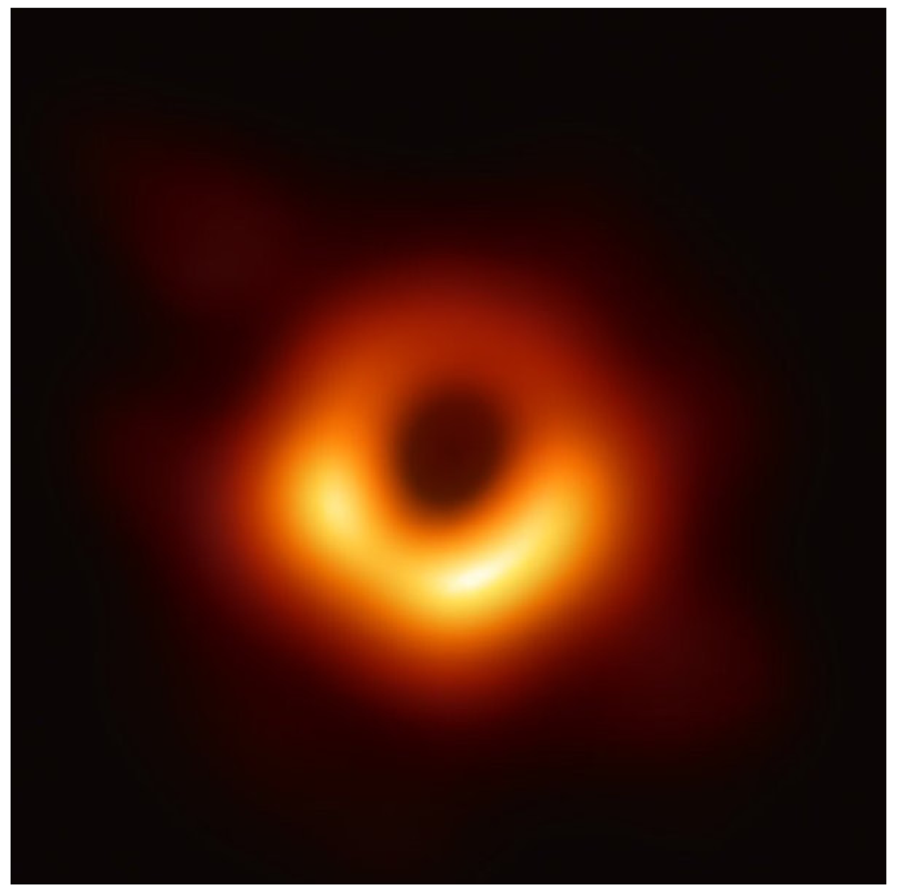

Figure 3: The first image of a black hole, exemplifying the current surge in observational black hole physics. [6]

Axisymmetry

The formulation I used was derived by invoking the Geroch decomposition followed by the ADM decomposition to yield a (2+1)+1 version of the Einstein field equations [7]. I assumed net zero angular momentum and initial data which is a superposition of scalar field initial data with a gravitational wave perturbation. This results in the following spatial metric

$$ds^2 = \psi^4 \left[e^{2\rho s} \left(d\rho^2 + dz^2 \right) + \rho^2 d\phi^2 \right]. \tag{1}$$

On the initial time slice one must solve for the conformal factor ψ using the Hamiltonian constraint

$$\partial_\rho^2 \psi + \frac{\partial_\rho \psi}{\rho} + \partial_z^2 \psi + \frac{1}{4}\psi \left(\rho \partial_\rho^2 s + 2\partial_\rho s + \rho \partial_z^2 s \right) + \frac{1}{4}\kappa \psi^5 e^{2\rho s} \rho_H = 0. \tag{2}$$

Similarly the lapse α may be computed using the maximal slicing condition $K = 0$

$$\partial_\rho^2 \alpha + \frac{\partial_\rho \alpha}{\rho} + \partial_z^2 \alpha + \partial_\rho \psi \partial_\rho \alpha + \partial_z \psi \partial_z \alpha = \frac{1}{2}\kappa \alpha e^{2\psi} e^{2\rho s}(\rho_H + \sigma + S). \tag{3}$$

By assuming time symmetric initial data, all other terms may be solved for trivially.

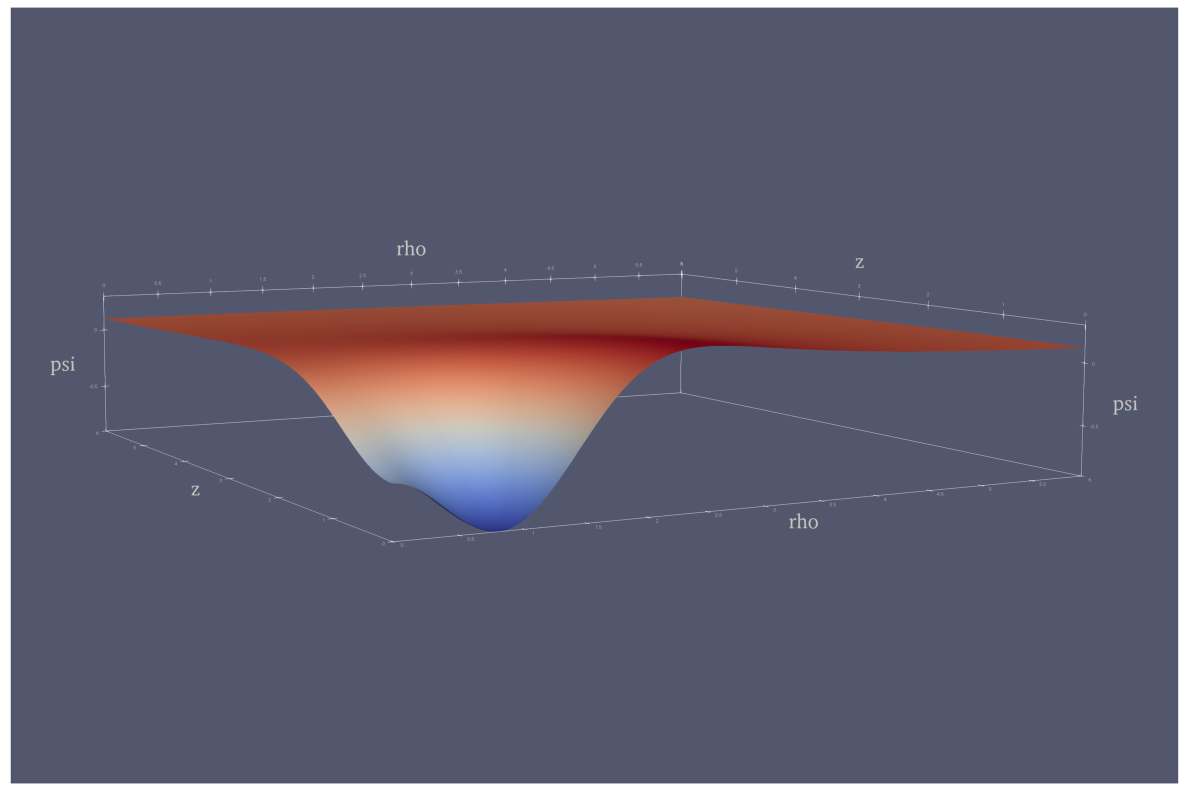

Figure 4: Solution for the conformal factor for vacuum gravitational wave data (also known as Brill waves). Here the Hamiltonian constraint is solved for ψ with $s = \rho^2 e^{-r^2}$ (the Gundlach seed function) and $\rho_H = 0$.

Numerical Methods

I wrote a finite difference code to solve the derived axisymmetric elliptic equations to 4th order. This code uses standard Lagrange basis polynomials to derive numerical stencils for value and derivative operators, and the resulting linear system is solved using a multigrid-accelerated, matrix-free algorithm. Block-structured Adaptive Mesh Refinement (AMR) is performed to balance computational work and preserve a high degree of numerical accuracy in regions with large gradients, where blocks are refined using a heuristic derived from the 8th order accurate solution to the Hamiltonian constraint. AMR is also a prerequisite for testing and observing echoing and self-similarity, as such effects usually occur on a spatio-temporal scale unattainable with a global mesh.

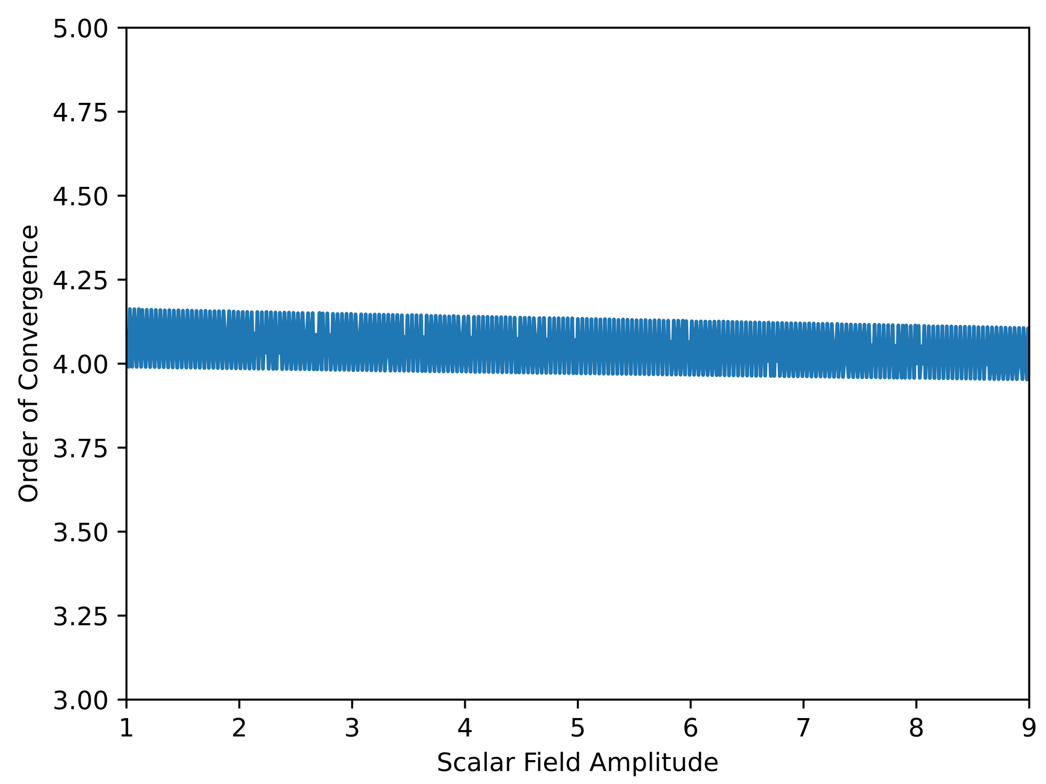

Figure 5: Approximate order of convergence, obtained by comparing ratio between error on 128x128 and 256x265 sized grids for initial scalar fields of varying amplitudes.

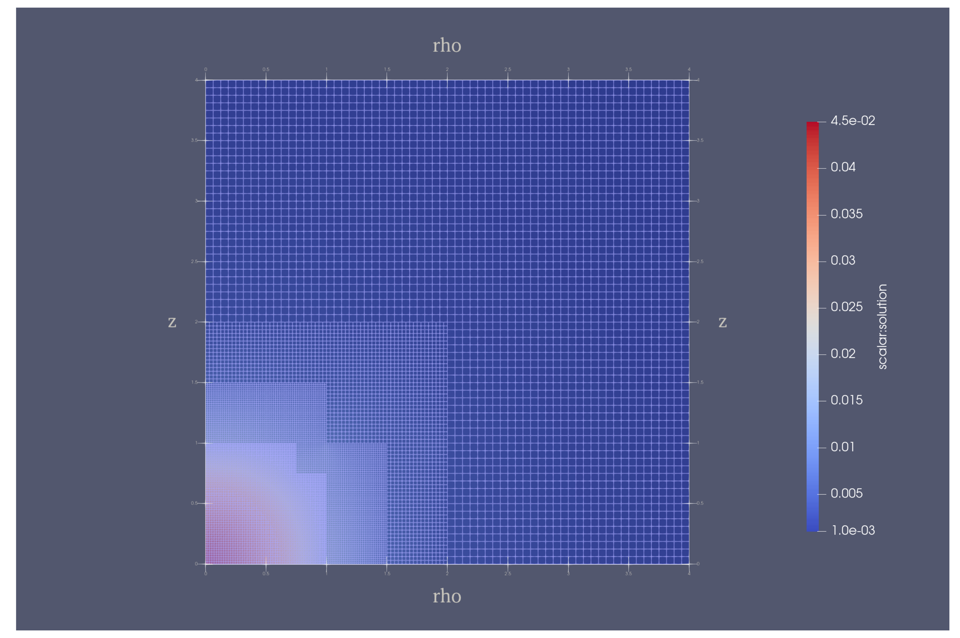

Figure 6: Example of AMR applied to scalar field initial data with no gravitational waves. Here four levels of refinement are used, and the Hamiltonian constraint for ψ is solved with $s = 0$ and $\rho_H = \frac{1}{2}\left(\phi^2 + \phi\right)$ where ϕ is a Gaussian scalar field.

Future Work

Now that I can solve the elliptic equations necessary for computing initial data and gauge variables, I am focusing on solving the remaining hyperbolic evolution equations. The code was also designed with GPU-acceleration in mind (and the core solver is GPU compatible), which will greatly improve performance of both the elliptic and hyperbolic solvers.

Acknowledgments

This project was funded in part by the University of Utah Physics & Astronomy Summer Undergraduate Research Program (SURP).

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